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Nakade — a graph theoretic concept in Go

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Abstract

We consider Go played on a given graph instead of on a checkered board. Let G be a graph. Let $G' = G + v$ be the join of G and another vertex v. Suppose that a white stone is placed at v in the initial stage. A graph G is called a nakade graph, if Black begins to play he can capture the white stone on v eventually. We investigate nakade graphs from a graph theoretic viewpoint. In particular, we focus on the length of smallest cycles, the maximum degree and the diameter of nakade graphs. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

Go is a game played with black and white stones on a checkered board by two players, Black and White. Black begins to play and each player plays alternately afterward. Each player chooses a vacant point on the board and places a stone of his color in his turn. One of the most fundamental rules of Go is the capturing rule. We shall note this rule for the game on a general graph F in terms of graph theory. Let F be a finite, non-directed graph without loops or multiple edges. Let $V(F)$ and $E(F)$ denote the vertex set and the edge set of F, respectively. For a vertex $x \in V(F)$, we denote the set of vertices adjacent to x by $N_F(x)$ or simply by $N(x)$ if the identity of F is clear from the context. For a vertex set $X \subseteq V(F)$, we denote $\bigcup_{x \in X} N(x)$ by $N(X)$. We denote two distinct players by P and Q.

Capturing rule. *Let $X \subseteq V(F)$ be occupied by Q's stones. If $N(X)\backslash X$ is occupied by P's stones after a P's move, then Q's stones on X are captured by P and removed from the board.*

If a P's move captures no stones, and leaves a set $Y \subseteq V(F)$ such that Y is occupied by P's stones and $N(Y)\backslash Y$ is occupied by Q's stones, then such move is called a suicidal move. Any suicidal move is banned.

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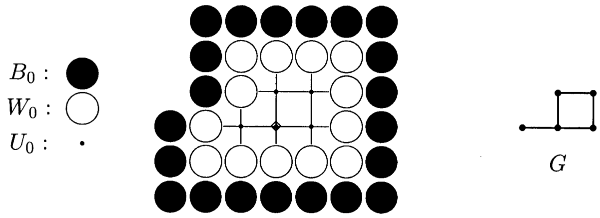

Fig. 1. If Black begins to take the marked point in U_0, he can eventually capture a group of white stones on W_0. Hence, G is a nakade graph.

When we play Go, our primary concern is whether a group of stones is alive or dead, namely it can be captured or not. In the following, we define the *Life-and-Death game*. Let F be a given graph. Let U_0, W_0 and $B_0 \subseteq V(F)$ be the unoccupied set, the set with white stones and the set with black stones in the initial stage, respectively. Roughly speaking, we consider a situation such that U_0 is enclosed by W_0, and W_0 is enclosed by B_0 (see Fig. 1).

Formally, we assume the following conditions:

(i) W_0 is connected.
(ii) $N(U_0)\backslash U_0 \subseteq W_0$.
(iii) $N(W_0)\backslash W_0 \subseteq U_0 \cup B_0$.
(iv) $N(u) \cap W_0 \neq \emptyset$ for any $u \in U_0$.

Both player's moves are restricted in U_0. Black begins to play. White may choose a pass instead of playing a stone on the board. The aim of Black is to capture white stones on W_0. An important remark is that if two players are competent, whether Black wins or not depends only on the graph $\langle U_0 \rangle_F$, the graph induced by U_0. We define $G = \langle U_0 \rangle_F$ as a *nakade* graph if Black can capture white stones on W_0. (Nakade is a Japanese Go term, which means 'a move from the inside'. In [5], a nakade graph appeared in the interior of the standard board is called a *big eye shape*.) We note some more remarks of this game.

- If White disconnects $\langle U_0 \rangle_F$ with white stones in the middle of the game, then White wins. Because if White continues to pass afterward, Black has no choice but to play a suicidal move in time.
- Condition (iv) implies that any white stone once placed on the board is not removed until the end of the game. On the other hand, black stones on U_0 may be captured by White. However, this means no game over. Black may continue the game on the region of his stones removed. In general, Black needs many sacrifices until he wins.
- If one of the conditions (i)–(iv) is dropped, the game becomes much more complicated even on the standard board. Most books for Go players deal with such problems. See [6], for example.

In their excellent book [5], Berlekamp and Wolfe shed light of combinatorial game theory on Go. Their theory is extremely useful at the end stage of Go. (The book contains a fruitful discussion on rules of Go as well.)

In this paper, we study Go from a graph theoretic viewpoint. In the following sections, we investigate the relationship between nakade graphs and their graph parameters. In Section 2, we describe basic facts on nakade graphs. In Sections 3 and 4, we focus on the maximum degree and the diameter of nakade graphs, respectively. In Section 5, we propose some open questions.

2. Basic facts

Before stating some basic facts on nakade graphs, we shall restate the definition of nakade graphs for our convenience.

Definition. The family of all *nakade* graphs $\mathcal{N}$ is defined as follows.

The graph with a single vertex K_1 is in $\mathcal{N}$. Let G be a connected graph with $n \geqslant 2$ vertices. Let us consider the *Life-and-Death game*, which is a vertex-coloring game on G by two players, Black and White, as follows:

- Black plays the first move and two players play alternately.
- Each player chooses a previously uncolored vertex and assigns it his color at his turn. Only White may pass instead of choosing a vertex.
- If White cuts G after some steps, that is, the deletion of white vertices disconnects G, then White wins and the game ends.
- If Black colors the last vertex, then Black wins.
- If White colors the last vertex, Black wins if and only if the subgraph induced by black vertices is a member of $\mathcal{N}$.

Then $G \in \mathcal{N}$ if and only if Black has a winning strategy for the game on G.

It is easy to see that the above definition is equivalent to that of Section 1. The Life-and-Death game is considered as a version of positional games where the first player wins if he establishes a winning configuration. Various games of this type are seen in Chapter 22 of [4]. One of the interesting features of our game is that a family of winning configurations $\mathcal{N}$ is not previously given, but recursively determined.

Example. Let P_n and C_n be the path and cycle with n vertices, respectively. Then P_n is nakade if and only if $n \leqslant 3$. Indeed, it is easy to see that P_1 and P_2 are nakade. For P_3, Black wins by taking the center in the first move. Hence, P_3 is nakade. For P_n with $n \geqslant 4$, White can cut the graph in the second move. Hence, P_n with $n \geqslant 4$ is not nakade. Similarly we find that C_n is nakade if and only if $n \leqslant 4$.

Fact 1. *If G is nakade then $G + e$ is nakade for any additional edge $e \notin E(G)$.*

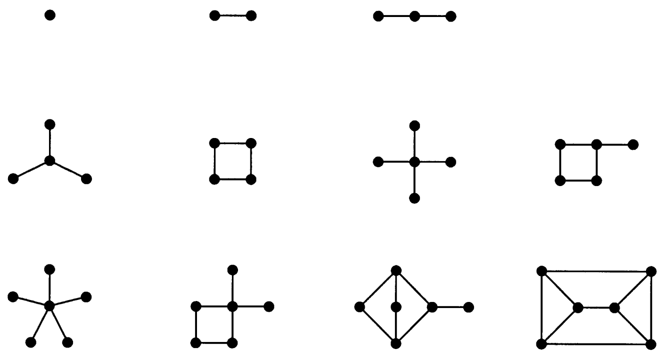

Fig. 2. Edge-minimal nakade graphs of order $\leqslant 6$.

Proof. For the game on $G + e$, Black can employ the same strategy for the game on G. $\square$

From Fact 1, it is essential to consider edge-minimal nakade graphs. Fig. 2 is a complete list of edge-minimal nakade graphs of order $\leqslant 6$.

Fact 2. *All complete bipartite graphs $K_{m,n}$ are nakade.*

Proof. We proceed by induction on the number of vertices. If $m = n = 1$, then $K_{1,1}$ is nakade. Let us consider the game on $K_{m,n}$ with $m \geqslant 2$. Black can take at least one vertex from each partite set so that the graph is not cut by White. When all the vertices are colored, the subgraph induced by black vertices is a complete bipartite graph of order $< m + n$, because White is forced to color at least one vertex. Hence, by the inductive hypothesis $K_{m,n}$ is nakade. $\square$

Fact 3. *If T is a tree, then T is nakade if and only if T is a star.*

Proof. If T is a star, T is nakade from Fact 2. If T is not a star, White can cut T in the second move. Hence, T is not nakade. $\square$

Fact 4. *Let G be a nakade graph. Then there exists a vertex $x \in V(G)$ such that $G - x$ is also nakade.*

Proof. For the game on G, assume that White continues to pass except in the forced turn at the end of the game. Let x be the vertex colored by White in the last turn. Then $G - x$ must be nakade to guarantee G is nakade. $\square$

The following result is derived from the above facts.

Theorem 2.1. *Let G be a nakade graph. If G is not a star, then G contains a cycle of length $\leqslant 4$.*

Proof. Assume that there is a nakade graph G which is not a star and contains no cycle of length $\leqslant 4$. By applying Fact 4 repeatedly, we have a sequence of nakade graphs $K_1 = G_1 \subset G_2 \subset \cdots \subset G_n = G$ such that $|V(G_i)| = i$ for each i and G_i is an induced subgraph of G_{i+1} for $1 \leqslant i \leqslant n-1$. Let j be the maximum index such that G_j contains no cycle. Since any cycle contained in G has length $\geqslant 5$, it follows that G_j is not a star. From Fact 3, G_j is not nakade, a contradiction. □

3. Maximum degree of nakade graphs

If a graph G contains a vertex x of large degree, it is often a good strategy for Black to occupy x in his first turn. Let $n(G)$ denote the number of vertices of G. Let $\Delta(G)$ denote the maximum degree of G. If $\Delta(G) = n(G) - 1$, G is always nakade from Fact 3 of Section 2. In the case $\Delta(G) = n(G) - 2$, there holds the following proposition.

Proposition 3.1. *Let $n \geqslant 4$. Let G_n be a graph with n vertices such that*

$$V(G_n) = \bigcup_{i=1}^{4} \{x_i\} \cup \bigcup_{i=1}^{n-4} \{y_i\}, \quad \textit{and}$$

$$E(G_n) = \{x_1x_2, x_2x_3, x_3x_4, x_4x_1\} \cup \bigcup_{i=1}^{n-4} \{x_1 y_i\}.$$

Then G_n is an edge-minimal nakade graph.

Proof. Black's strategy is as follows. Black chooses x_1 in his first turn. If White chooses x_2(or x_4), then Black chooses x_4(or x_2). Moreover, Black can force White to choose x_3 at the final turn. Then the subgraph induced by black vertices is a star, which is nakade. Hence G_n is nakade. For any edge $e \in E(G_n)$, $G_n - e$ is either a disconnected graph or a tree other than a star. Therefore, $G_n - e$ is not nakade from Fact 3 in Section 2. It follows that G_n is an edge-minimal nakade graph. □

Let $f(n)$ be the minimum integer d such that there exists a nakade graph G with $n(G) = n$ and $\Delta(G) = d$. The aim of this section is to estimate $f(n)$. First, we focus on bipartite graphs. Let $f_b(n)$ be the minimum integer d such that there exists a bipartite nakade graph G with $n(G) = n$ and $\Delta(G) = d$. The base of logarithm is two throughout the paper.

Theorem 3.2.

$$\frac{n}{2} - \log n + \mathrm{O}(1) \leqslant f_b(n) \leqslant \frac{n}{2} - \frac{\log n}{2} + \mathrm{O}(1).$$

We use the following two lemmas to prove Theorem 3.2.

Lemma 3.3. *Let r be a non-negative integer. Let G be a bipartite graph such that*

$$V(G) = \bigcup_{i=0}^{r} X_i \cup \bigcup_{i=0}^{r} Y_i,$$

where $|X_0| \geqslant 1$, $|Y_0| \geqslant 2^r$ *and* $|X_i| \geqslant 2^{i-1}$, $|Y_i| \leqslant 1$ for $1 \leqslant i \leqslant r$, *and*

$$E(G) = \bigcup_{i=1}^{r} \{xy\colon x \in X_i, y \in Y_i\} \cup \bigcup_{i=0}^{r} \{xy\colon x \in X_i, y \in Y_0\}.$$

Then G is nakade.

Proof. We proceed by induction on r. If $r = 0$ then G is complete bipartite, and so it is nakade from Fact 2 in Section 2. Assume $r \geqslant 1$. Black's strategy is as follows.

(1) Choose a vertex of X_1 in the first move.
(2) If White chooses a vertex X_i for $2 \leqslant i \leqslant r$ or Y_0, and if there remains an uncolored vertex v in the same set, then choose v so that at least $\lfloor |X_i|/2 \rfloor$ or $\lfloor |Y_0|/2 \rfloor$ vertices of the set are colored black.
(3) If $Y_1 \neq \emptyset$, never choose Y_1 and force White to choose Y_1 in the last turn.

If Black follows this strategy, White cannot cut G. Moreover, when all vertices are colored, the graph induced by black vertices also satisfies the assumption of the lemma with r decreased. Thus, the assertion follows from the inductive hypothesis. □

Lemma 3.4. *Let G be a graph. Let X be an independent set of G. Let $Y = V(G)\backslash X$. If there exists a non-negative integer r such that $|Y| \leqslant 2^r$ and $|X| - |X \cap N(y)| \geqslant r+1$ for any $y \in Y$, then G is not nakade.*

Proof. We proceed by induction on r. If $r = 0$ then G is not connected. Hence, it is not nakade. Assume $r \geqslant 1$. White's strategy is choosing vertices in Y as many as possible and leaving vertices in X as many as possible. Let X' and Y' be sets of black vertices in X and Y at the end of the game, respectively. Let G' be the graph induced by $X' \cup Y'$. Then, we have $|Y'| \leqslant 2^{r-1}$. On the other hand, since at most one vertex in X is colored white, we have $|X'| - |X' \cap N_{G'}(y)| \geqslant r$ for any $y \in Y'$. Now the inductive hypothesis completes the proof. □

Proof of Theorem 3.2. *Upper bounds.* For a given n, let us take the maximum integer r such that $2^{r+1} + r \leqslant n$. Let $s = n - (2^{r+1} + r)$. We consider a nakade graph G with n vertices in Lemma 3.3 with $|X_0| = 1 + \lceil s/2 \rceil$, $|Y_0| = 2^r + \lfloor s/2 \rfloor$ and $|X_i| = 2^{i-1}$, $|Y_i| = 1$ for $1 \leqslant i \leqslant r$. Then, we have

$$\begin{aligned} \Delta(G) &= 2^r + \lfloor s/2 \rfloor + 1 \\ &= 2^r + \lfloor (n - 2^{r+1} - r)/2 \rfloor + 1 \\ &\leqslant \frac{n}{2} - \frac{\log n}{2} + \frac{5}{2}. \end{aligned}$$

Lower bounds. Let G be a bipartite graph with two partite sets X, $Y \subseteq V(G)$ such that $|Y| \leqslant |X|$. Suppose that $\Delta(G) \leqslant n/2 - \log n$. Let r be the minimum integer such that $|Y| \leqslant 2^r$. Then, we have

$$|X| - \Delta(G) \geqslant \frac{n}{2} - \left(\frac{n}{2} - \log n\right)$$
$$> r \quad (\text{since } n > 2^r).$$

It follows from Lemma 3.4 that G is not nakade. This completes the proof. □

Next we consider $f(n)$. Since $f(n) \leqslant f_b(n)$ holds, we have $f(n) \leqslant n/2 - (\log n)/2 + \mathrm{O}(1)$. Until now we have no better upper bound. We have the following result for lower bounds.

Theorem 3.5. *There exists an absolute constant $c > 0$ such that $f(n) > cn$.*

We first show an asymptotically weaker result than Theorem 3.5.

Lemma 3.6. $f(n) > \sqrt{n} - \log n$ *for* $n \geqslant 2$.

Proof. We may assume $n \geqslant 4$. Let G be a graph with n vertices and the maximum degree d. Assume $d \leqslant \sqrt{n} - \log n$. We employ Lemma 3.4 to show that G is not nakade. Let X be a maximal independent set of G. Let $Y = V(G) \setminus X$. Since $|X| \geqslant \lceil n/(d+1) \rceil \geqslant \sqrt{n} + 1$, we have $|X| - \Delta(G) \geqslant \lceil \log n \rceil + 1$. From Lemma 3.4 with $r = \lceil \log n \rceil$, it follows that G is not nakade, as required. □

For the proof of Theorem 3.5, we employ the 'probabilistic method', which has turned out to be very effective in analyzing combinatorial games [2,3]. Let V be a finite set. Let $\mathscr{A}$ be a family of subsets of V. Let us define the *discrepancy* of $\mathscr{A}$ with respect to a subset $X \subseteq V$ by

$$\operatorname{disc}(\mathscr{A}, X) = \max_{A \in \mathscr{A}} ||A \cap X| - |A \cap X^c||,$$

where $X^c = V \setminus X$, and the discrepancy of $\mathscr{A}$ by

$$\operatorname{disc}(\mathscr{A}) = \min_{X \subseteq V} \operatorname{disc}(\mathscr{A}, X).$$

The following lemma is essentially Theorem 1.1 in Chapter 12 of [1].

Lemma 3.7. *Let $\mathscr{A}$ be a family of m subsets of a finite set V with $|A| \leqslant d$ for any $A \in \mathscr{A}$. Then*

$$\operatorname{disc}(\mathscr{A}) \leqslant \sqrt{2d \log(2m)}. \qquad \square$$

We rewrite Lemma 3.7 for our purpose as follows.

Lemma 3.8. *Let G be a graph with $n(G)=n$ and $\Delta(G)=d$. Then there exist an absolute constant c_0 and a partition of vertices $V(G)=X\cup Y$ such that*

$$\max_{v\in V(G)}\max\{|N(v)\cap X|,|N(v)\cap Y|\}\leqslant\frac{d}{2}+c_0\sqrt{d\log n}.$$

Proof. Let $V=V(G)$ and $\mathscr{A}=\{N(v)\}_{v\in V(G)}$, and apply Lemma 3.7. $\square$

Proof of Theorem 3.5. First we list constants used in the proof.

$c_0=$ the constant in Lemma 3.8.

$c_1=\left(\frac{1}{2}\right)^{7/8}-\frac{1}{2}.$

$c_2=(\sqrt{2}\,c_0+1)/c_1.$

$N_0=\min\{n\colon 1+m^{3/8}\leqslant m^{1/2}-\log m \text{ for any } m\geqslant n\}.$

$N_1=\min\{n\colon c_2\log m\leqslant m^{1/8} \text{ for any } m\geqslant n\}.$

$N=\max\{2N_0,N_1\}.$

$a=N^{-1}.$

$b=N^{-1/2}.$

In the following, we shall prove the claim which asserts if a graph G with n vertices satisfies $\Delta(G)\leqslant an+bn^{7/8}$ then G is not nakade for $n\geqslant N_0$. The theorem follows from this claim easily. We prove the claim by induction on n.

Case 1. $N_0\leqslant n<N$: In this case, we have

$$\begin{aligned}\Delta(G)&\leqslant an+bn^{7/8}\\&\leqslant 1+n^{3/8}\\&\leqslant n^{1/2}-\log n.\end{aligned}$$

From Lemma 3.6, it follows that G is not nakade.

Case 2. $N\leqslant n$: Let $V(G)=X\cup Y$ be a partition satisfying the inequality of Lemma 3.8. We may assume $|X|\leqslant|Y|$. Let $X_1=X$, $Y_1=Y$ and $G_1=G$. We recursively define $X_i\subseteq X$, $Y_i\subseteq Y$ and a graph $G_i=\langle X_i\cup Y_i\rangle_G$ for $i>1$ as follows. Let us consider the game on G_i. White's strategy is to choose an uncolored vertex of X_i with the following exceptions.

(1) If it is the last turn, choose the remaining vertex.
(2) If it is not the last turn, and there remains no uncolored vertex in X_i, choose a passing.
(3) If it is not the last turn, and the sum of the number of black vertices and the number of uncolored vertices decreases to $\lceil n/2\rceil+1$, then choose a passing.

Let X_{i+1} and Y_{i+1} be a set of black vertices in X_i and Y_i at the end of the game on G_i, respectively. Then, White's strategy guarantees that $|Y_{i+1}| \geqslant |Y_i| - 1$ and $|X_{i+1}| \leqslant \lceil |X_i|/2 \rceil$ if (3) does not occur. Let $G_{i+1} = \langle X_{i+1} \cup Y_{i+1} \rangle_G$. If $|X_{i+1}| > 1$ and $\lceil n/2 \rceil < n(G_{i+1})$, then we continue the game on G_{i+1}, else we set $H = G_{i+1}$ and calculate $\Delta(H)$. We have $n(H) \geqslant \lceil n/2 \rceil \geqslant N_0$. We claim $|X_{i+1}| \leqslant \log n$. Indeed, if $|X_{i+1}| > 1$, we have

$$\begin{aligned}
|X_{i+1}| &= \lceil n/2 \rceil - |Y_{i+1}| \\
&\leqslant \lceil n/2 \rceil + i - |Y| \quad (\text{since} |Y_{j+1}| \geqslant |Y_j| - 1 \text{ for } 1 \leqslant j \leqslant i) \\
&\leqslant i \quad (\text{since } |X| \leqslant |Y|) \\
&\leqslant \log n \quad (\text{since } |X_{j+1}| \leqslant \lceil |X_j|/2 \rceil \text{ for } 1 \leqslant j \leqslant i-1).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\Delta(H) &\leqslant \max_{v \in V(H)} |N_H(v) \cap Y| + |X_{i+1}| \\
&\leqslant \frac{\Delta(G)}{2} + c_0 \sqrt{\Delta(G) \log n} + \log n \quad (\text{from Lemma 3.8}).
\end{aligned}$$

By using the list of the constants, we have

$$\begin{aligned}
&an(H) + bn(H)^{7/8} - \Delta(H) \\
&\geqslant a\frac{n}{2} + b\left(\frac{n}{2}\right)^{7/8} - (an + bn^{7/8})/2 - c_0 \sqrt{(an + bn^{7/8}) \log n} - \log n \\
&\geqslant c_1 bn^{7/8} - c_0 \sqrt{2bn \log n} - \log n \\
&\geqslant c_1 bn^{7/8} - (\sqrt{2}\, c_0 + 1)\sqrt{bn} \log n \\
&= c_1 \sqrt{bn}(b^{1/2} n^{3/8} - c_2 \log n) \\
&\geqslant c_1 \sqrt{bn}(n^{1/8} - c_2 \log n) \\
&\geqslant 0.
\end{aligned}$$

Since $n(H) \geqslant N_0$, it follows from the inductive hypothesis that H is not nakade. Therefore, G is not nakade, as required. □

4. Diameter of nakade graphs

In this section, we investigate the diameter of nakade graphs. Let x and y be two vertices of a given graph G. The *distance* of x and y is the length of the shortest path from x and y in G, which is denoted by $d_G(x, y)$. The *diameter* of G is defined as $\operatorname{diam}(G) = \max\{d_G(x, y)\colon \{x, y\} \subseteq V(G)\}$. For a positive integer d, we define $g(d)$ as the smallest integer n such that there exists a nakade graph G with $n(G) = n$ and $\operatorname{diam}(G) = d$. We find $g(1)=2$, $g(2)=3$, $g(3)=5$ and $g(4)=9$ by simple calculations. For positive integers k and l, let $tow_k(l)$ denote the tower function, that is, $tow_k(l)=2^{2^{\cdot^{\cdot^{\cdot^{2^l}}}}}$ such that 2 appears $k-1$ times.

Theorem 4.1. *For* $d\geqslant 1$,

$$tow_{\lfloor d/3\rfloor+1}(1) < g(d) < tow_{\lfloor d/3\rfloor+1}(5).$$

First we show a lemma used to prove upper bounds of $g(d)$.

Lemma 4.2. *Let d be a positive integer. Let $\{X_i\colon 0\leqslant i\leqslant d\}$ be a family of mutually disjoint non-empty sets. Let G be a graph such that*

$$V(G)=\bigcup_{i=0}^{d} X_i,$$

$$E(G)=\bigcup_{i=0}^{d-2}\{uv\colon u\in X_i, v\in X_{i+2}\}\cup\{uv\colon u\in X_{d-1}, v\in X_d\}.$$

Let $x_i=|X_i|$ for $0\leqslant i\leqslant d$. If

$$x_i\geqslant 2^{\sum_{j\leqslant i-3}x_j}$$

holds for any $3\leqslant i\leqslant d$, then G is nakade.

Proof. We proceed by induction on $n=n(G)$. If $d\leqslant 2$, G is complete bipartite and nakade. Assume $d\geqslant 3$. Black's strategy for the game on G is as follows.

(1) Choose a vertex in X_2 in the first move.
(2) Choose a vertex in the same set as White chose just before, if possible.
(3) Force White to choose a vertex in X_0 in the last move.

Then, White cannot disconnect G, because $x_i\geqslant 2$ for any $i\geqslant 3$. Let G' be the subgraph induced by black vertices at the end of the game. Let $X_i'=X_i\cap V(G')$ and $x_i'=|X_i'|$ for each i. Then, we have

$$\begin{aligned} x_i' &\geqslant \lfloor x_i/2\rfloor \\ &\geqslant 2^{\sum_{j\leqslant i-3} x_j-1} \\ &\geqslant 2^{\sum_{j\leqslant i-3} x_j'}, \end{aligned}$$

for any $i\geqslant 3$. Hence G' is nakade by the inductive hypothesis. Thus G is also nakade. □

Next we show a lemma used to prove lower bounds of $g(d)$.

Lemma 4.3. *Let G be a graph. Suppose that there exists a partition of vertices $V(G)=\bigcup_{i=0}^{d}X_i$* with *$X_i\neq\emptyset$ for $0\leqslant i\leqslant d$ such that $E(G)\subseteq\bigcup_{i=0}^{d-1}\{uv\colon u\in X_i,$*

$v \in X_{i+1}\}$. Let $x_i = |X_i|$ for $0 \leqslant i \leqslant d$. Furthermore, we set $l_i = \sum_{j<i} x_j$ and $r_i = \sum_{i<j} x_j$ for each i. If there exists a pair of indices $\{i,j\}$ with $0 < i < j < d$ such that

$$\max\{x_i, x_j\} \leqslant 2^{\min\{l_i, r_j\}-1},$$

then G is not nakade.

Proof. We proceed by induction on $p = \min\{l_i, r_j\}$, where i and j are two indices satisfying the inequality of the lemma. If $p = 1$, it follows that $x_i = x_j = 1$ and White can cut G in the second move. Thus G is not nakade. Assume $p \geqslant 2$. White's strategy is to choose vertices in $X_i \cup X_j$ as many as possible. Let G' be the graph induced by black vertices at the end of the game. Let $X'_k = V(G') \cap X_k$, $x'_k = |X'_k|$, $l'_k = \sum_{l<k} x'_l$ and $r'_k = \sum_{k<l} x'_l$ for $0 \leqslant k \leqslant d$. Then, we have

$$\begin{aligned} \max\{x'_i, x'_j\} &\leqslant \max\{\lceil x_i/2 \rceil, \lceil x_j/2 \rceil\} \\ &\leqslant 2^{\min\{l_i, r_j\}-2} \\ &\leqslant 2^{\min\{l'_i, r'_j\}-1}. \end{aligned}$$

Hence, G' is not nakade by the inductive hypothesis. Thus G is not nakade. □

Proof of Theorem 4.1. *Upper bounds.* We define a sequence of integers $\{a_i\}_{0 \leqslant i}$ such that

$$a_i = \begin{cases} 1 & \text{for } 0 \leqslant i \leqslant 2, \\ 2^{\sum_{j=0}^{i-3} a_j} & \text{for } i \geqslant 3. \end{cases}$$

From Lemma 4.2 there exists a nakade graph G of order $n = \sum_{i=0}^{d} a_i$ with $\mathrm{diam}(G) = d$. It suffices to prove the required inequality for $d \equiv 2 \bmod 3$. Let $d = 3t - 1$.

Claim 1. *$a_{3t-1} \leqslant \frac{1}{4} tow_t(5)$ for any $t \geqslant 1$.*

If $t \leqslant 2$, the inequality follows from $a_2 = 1$ and $a_5 = 8$. Let $t \geqslant 3$. Then, we have

$$\begin{aligned} a_{3t-1} &= 2^{\sum_{i=0}^{3t-4} a_i} \\ &\leqslant 2^{3 \sum_{j=1}^{t-1} a_{3j-1}} \\ &\leqslant 2^{\frac{3}{4} \sum_{j=1}^{t-1} tow_j(5)} && \text{(by induction)} \\ &\leqslant 2^{tow_{t-1}(5)-2} && \text{(by easy calculation)} \\ &= \frac{1}{4} tow_t(5). \end{aligned}$$

Thus Claim 1 is proved.

Since $n = \log a_{3t+2}$, we have $n < tow_t(5)$ from Claim 1. This implies $g(d) < tow_{\lfloor d/3 \rfloor + 1}(5)$.

Lower bounds. It is easy to see that the required inequality holds for $d \leqslant 5$. We assume $d \geqslant 6$. Let G be a nakade graph with n vertices and diameter d. Let x be a vertex of G with $d_G(x,v) = d$ for some vertex $v \in V(G)$. Let $X_i = \{v\colon d_G(x,v) = i\}$ and $x_i = |X_i|$ for $0 \leqslant i \leqslant d$. Let us define a family of disjoint sets of indices $\{A_p\}_{0 \leqslant p}$ such that

$$A_0 = \{i \in [0,d] : x_i = 1\},$$

$$A_p = \{i \in [0,d] : tow_p(1) < x_i \leqslant tow_{p+1}(1)\} \quad \text{if } p \geqslant 1.$$

Claim 2. $|A_p| \leqslant 3$ *for any* $p \geqslant 0$.

If $|A_0| \geqslant 4$, White can cut G in the second move, a contradiction. Assume $|A_p| \geqslant 4$ for some $p \geqslant 1$. Let $i < j < k < l$ be four distinctive indices in A_p. Then, we have

$$\begin{aligned}\max\{x_j, x_k\} &\leqslant tow_{p+1}(1)\\ &\leqslant 2^{\min\{x_i,x_l\}-1}.\end{aligned}$$

From Lemma 4.3, G is not nakade, a contradiction.

We shall give a better estimation of $|A_p|$ for small p. Let $A' = A_0 \cup A_1 \cup A_2$. Note that $A' = \{i \in [0,d]\colon 1 \leqslant x_i \leqslant 4\}$.

Claim 3. $|A'| \leqslant 6$.

To the contrary, assume $\{i_1 < i_2 < i_3 < i_4 < i_5 < i_6 < i_7\} \subseteq A'$. White's strategy is to choose vertices in X_{i_3}, X_{i_4} and X_{i_5}. Let $G_1 = G$. Let G_{j+1} be the graph induced by black vertices at the end of the game on G_j for $j \geqslant 1$. At the beginning of the game on G_3, each X_i for $i = i_3, i_4, i_5$ is reduced to at most one vertex, and at least one vertex remains in $X_{i_1} \cup X_{i_2} \cup X_{i_6} \cup X_{i_7}$. Hence, White can cut G_3. This is a contradiction, as required.

By Claims 2 and 3, we have $A_{\lfloor d/3 \rfloor + 1} \neq \emptyset$ if $d \geqslant 6$. It follows that $n \geqslant \max x_i > tow_{\lfloor d/3 \rfloor + 1}(1)$, as claimed. □

5. Open questions

In this section, we summarize some open questions related to the concepts of the paper. First let us recall that $f(n)$ is the minimum integer d such that there is a nakade graph G with $n(G) = n$ and $\Delta(G) = d$. From Theorems 3.2 and 3.5, we have $cn < f(n) \leqslant n/2 - \log n/2 + \mathrm{O}(1)$ with a positive constant c for large n.

Question 1. What is an approximate value of $f(n)$ when n tends to infinity?

The opposite question of Question 1 may be considered. Let $h(n)$ be the greatest integer $d < n$ such that any graph with $n(G) = n$ and the minimum degree $\delta(G) = d$ is a nakade graph. In order to give a lower bound of $h(n)$, we construct a graph G as follows: $V(G) = \bigcup_{i=1}^{5} V_i$, where $V_i \neq \emptyset$ for any i and $|V_i| - |V_j| \leqslant 1$ for any i and j, and $E(G) = \{uv: u \in V_i,\ v \in V_j$ such that $j - i \equiv 0, 1 \pmod 5\}$. It turns out that G is not a nakade graph. It follows that $\lfloor 3n/5 \rfloor \leqslant h(n)$.

Question 2. What is an approximate value of $h(n)$ when n tends to infinity?

The last question corresponds to a counting problem of nakade graphs. Until now we have no idea whether the number of nakade graphs is larger than that of non-nakade graphs.

Question 3. What is an approximate order of nakade graphs with n vertices?

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